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THE VARIATIONAL PROBLEM OF THE ONE-DIMENSIONAL ISENTROPIC COMPRESSION OF AN IDEAL GAS†

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The variational problem of the one-dimensional shock-free compression of an ideal (non-viscous and non-heat conducting) gas by a plane ($\nu=0$) cylindrical ($\nu=1$) and spherical ($\nu=2$) piston is considered. As in [1, 2], the work of the piston is minimized for a specified displacement in a fixed time t_f. The time τ_0 taken by a sound wave to traverse the section x_a–$x_{a°}$ plays an important role in the formulation of the problem. Here x is the Cartesian, cylindrical or spherical coordinate, and x_a and $x_{a°}$ correspond to the piston (at $t=0$) and the fixed wall (for $\nu=1$ and 2, possibly, the axes or the centre of symmetry). Unless otherwise stated, $x_{a°}<x_a$, and the piston in the xt plane moves to the left. In formulating the problem in a gas when $t<t_f$ no shockwaves are permitted. Hence, if $t_f<\tau_0$, then to the left of the "initial" c^--characteristic the gas is unperturbed and can be eliminated from consideration, i.e. the case $t_f<\tau_0$ reduces to the case $t_f=\tau_0$ with smaller τ_0 and larger $x_{a°}$. Unlike [1, 2], where the gas at $t=0$ was assumed to be at rest and uniform, henceforth for the zeroth x-component of the velocity, variability of the initial entropy is allowed, while for $\nu=1$ a radially equilibrium initial twist is permitted.

ONLY THE case $t_f<\tau_0$ was considered in [1, 2], for which, when $\nu=0$, the problem was solved exactly, while when $\nu\neq 0$ it was solved approximately (using plane flow of the "simple wave" type). The time t_f can have any value below. For $t_f=\tau_0$ an exact solution can be obtained for all ν by the undetermined control contour method [3]. Here we mean by an exact solution the reduction of the initial problem of constructing the optimum trajectory of the piston to the numerical solution of certain problems of one-dimensional non-stationary gas dynamics using the method of characteristics. In one of the problems solved by the method of characteristics the distribution of the parameters at the final part of the "extremal" c^+-characteristic, arriving at the end point of the piston trajectory is known. The condition for an extremum, which defines this part, turned out to be the same as in the problem of the optimum expansion of a piston [4].

In the case when $t_f>\tau_0$, the conditions for one other problem solved by the method of characteristics are imposed on the section of the horizontal $t=t_f$ close to the fixed wall, where the gas is either at rest or (when there is a twist for $\nu=1$) is radially at equilibrium. At a certain time $t_f=\tau_m>\tau_0$ it is at rest or the whole optimally compressed gas is in radial equilibrium. For $t_f>\tau_m$ this compression (with the same work) can be achieved in an infinite number of ways. The optimum compression when $t_f\geqslant\tau_m$ requires considerably less work than when $t_f=\tau_0$.

1. Suppose that, at the initial instant $t=0$, an ideal gas is enclosed in a plane cylindrical or spherical volume: $x_{a°}\leqslant x\leqslant x_a$. Henceforth, as a rule, the subscripts a, $a°$, ... will be attached to parameters at the points a, $a°$, ... in the xt plane (Fig. 1). We will write a zero subscript for the

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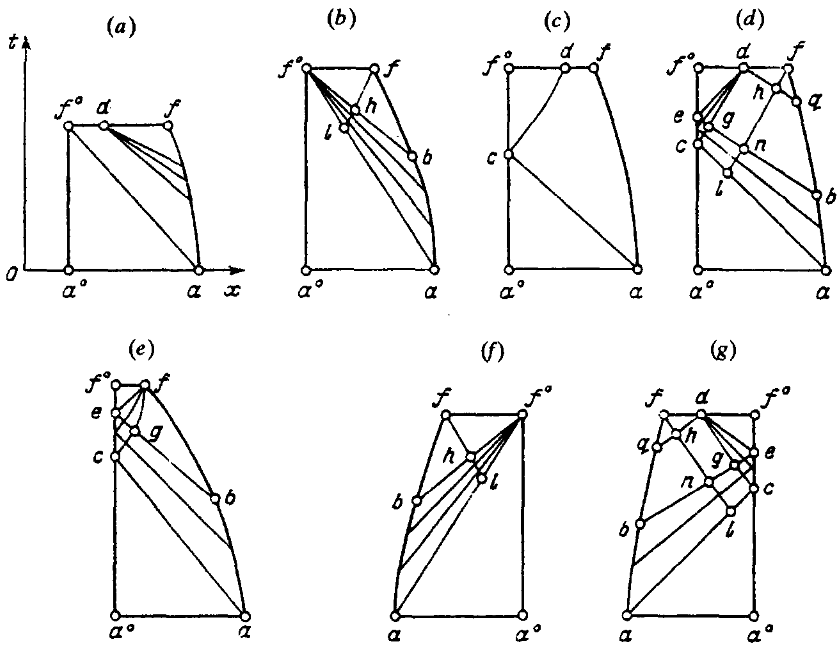

FIG. 1.

initial distributions of the parameters. In general, they can be functions of x. In particular, an arbitrary initial non-uniformity of the specific entropy $s_0(x)$ is permitted. The variability of the initial pressure p_0 will henceforth be allowed only for $\nu = 1$ as a consequence of the twisting of the flow—the circumferential component $\upsilon_0(x)$ of the velocity of the gas is non-zero. Its x-component u at the instant $t = 0$ is assumed to be zero for all ν while for $\nu = 1$ and $\upsilon_0(x) \not\equiv 0$ it is assumed that $p_0(x)$ satisfies the "radial equilibrium" condition

$$\partial p_0 / \partial x = \rho_0 \upsilon_0^2 / x \tag{1.1}$$

where ρ is the density of the gas. The thermodynamic parameters other than p and s (ρ, the specific volume $\omega = 1/\rho$, the specific internal energy e, the enthalpy $h = e + p/\rho$, the absolute temperature T, the velocity of sound a, etc.) are assumed to be specified functions of p and s, where

$$h = h(p,s), \quad \omega = \omega(p,s) = h_p, \quad T = T(p,s) = h_s$$

$$a^{-2} = a^{-2}(p,s) = \rho_p = -\omega_p / \omega^2 = -h_{pp} / h_p^2, \quad h_{ppp} = \omega_{pp} > 0 \tag{1.2}$$

Here $h(p,s)$ is a known function of p and s, and the subscripts p and s denote corresponding partial differentiation, the expressions for ω and T are a consequence of the thermodynamic equation: $Tds = dh - \omega dp$, while the inequality $\omega_{pp} > 0$ represents the definition of a "normal" gas. Only when it is satisfied will the piston, moving in the gas, form a compression wave, on which the characteristics travelling from the piston may intersect [5, 6]. In this sense the inequality from (1.2), i.e. a consideration solely of gases, called "normal" above, is henceforth fundamental.

The subsequent investigation also remains true for non-zero (and even x-dependent) initial distributions of the projections of the velocity vector on the y and z axes for $\nu = 0$ and of the axial component of the velocity for $\nu = 1$. These components, "preserved in the particle", have no effect on the remaining parameters. When $\nu = 2$ there are no components of the velocity vector differing from u, in view of the assumption of spherical symmetry.

If at the instant $t=0$, the piston, previously at rest, begins to move in an ideal gas, then, under the above-mentioned conditions in the gas, unsteady flow occurs with plane, cylindrical or spherical waves, defined by the tangency condition on the walls and on the piston and by the equations

$$\frac{dp}{dt}+\rho a^2\frac{\partial p}{\partial x}+\nu\frac{\rho u a^2}{x}=0,\quad \frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}-\nu(2-\nu)\frac{\upsilon^2}{x}=0$$
$$\frac{ds}{dt}=0,\quad \frac{d\Gamma}{dt}=0\quad\left(\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x},\Gamma=x^\nu\upsilon\right)\tag{1.3}$$

The first of these equations is a consequence of the equation of continuity

$$\partial(x^\nu\rho)/\partial t+\partial(x^\nu\rho u)/\partial x=0\tag{1.4}$$

converted taking (1.2) and the third equation of the same system into account. The second equation of (1.3) is the projection of the equation of motion on the x axis. The free term in it is only important when $\nu=1$, which is ensured by the factor $\nu(2-\nu)$. The third and fourth equations of (1.3) are the conditions for conservation of entropy and "angular momentum" Γ in a particle. The latter is only important for $\nu=1$.

The equation of continuity (1.4) allows us to introduce the Lagrange variable ψ, such that along any curve in the xt plane

$$d\psi=Kx^\nu\rho(dx-udt)\tag{1.5}$$

with an arbitrary normalizing factor K, which we shall henceforth take as positive. According to (1.5), ψ is constant along the particle trajectories (in particular, along the trajectories of the fixed wall $x=x_{a^0}$ and the piston). The introduction of ψ enables us to integrate the last two equations of (1.3)

$$s=s_0(\psi),\quad x^\nu\upsilon\equiv\Gamma=\Gamma_0(\psi)\tag{1.6}$$

The first integral (the "entropy integral") only holds when there are no shock waves in the flow considered. We will henceforth assume this, unless otherwise stated.

The first two equations of (1.3) can be replaced by characteristic equations containing derivatives only along the c^+- and c^--characteristics, respectively. Together with the equations of the characteristics in the xt and ψt planes they are equivalent to the following equations

$$dx=(u\pm a)dt,\quad d\psi=\pm Kx^\nu\rho a dt$$

$$du\pm\frac{1}{\rho a}dp\pm\frac{\nu}{x}[au\mp(2-\nu)\upsilon^2]dt=0\tag{1.7}$$

Here the upper (lower) signs correspond to c^+ (c^-)-characteristics. According to the second equation of (1.7), for movement along the c^+ (c^-)-characteristic in the direction in which t increases, the Lagrange variable ψ increases (decreases) monotonically.

The work A done by the piston when it moves for a time t_f from point a to point f, the coordinate of which $x_f<x_a$, apart from a positive factor that is unimportant for the variational problem, is equal to

$$A=-K\int_0^{t_f}px^\nu u dt=-K\int_{x_a}^{x_f}px^\nu dx\tag{1.8}$$

The factor $K>0$, the same as in (1.5), is introduced here for convenience.

We will formulate the variational problem. For initial parameters of the gas as specified above with $t=0$, $x_{a^\circ}\leqslant x\leqslant x_a$ and a fixed "internal" wall $x\equiv x_{a^\circ}\geqslant 0$ (in the xt plane its trajectory is the vertical $a^\circ f^\circ$) it is required to obtain the motion of the piston from the fixed point a: $x=x_a$, $t=0$ to the fixed point f: $x=x_f=x_a$, $t=t_f$, i.e. the dependence of the velocity of the piston u in (1.8) on t or on x, so that for shock-free flow for $t<t_f$ the work A is a minimum. The requirement of shock-free conditions indicates, in particular, that $u_a=0$, while the region of perturbed flow in the xt plane is bounded below by the (c^-)-characteristic, moving from point a. Since $u_0(x)\equiv 0$, the time τ_0 mentioned above, according to (1.7) is given by

$$\tau_0=-\int_a^{a^\circ}\frac{dx}{a_0(x)}=-\int_a^{a^\circ}\frac{d\psi}{Kx^\nu\rho_0(\psi)a_0(\psi)}$$

As stated earlier, here $t_f\geqslant\tau_0$, while in [1, 2], the case $t_f=\tau_0$ is considered for $a_0\equiv\text{const}$.

When solving the above variational problem, in addition to (1.8) we will use expressions for A which are consequences of the integral law of conservation of energy. The latter, together with the integral law of conservation of mass, can be written as

$$\begin{aligned}&K\oint\left\{x^\nu\rho\left[\frac{u^2}{2}+h+\nu(2-\nu)\frac{\Gamma^2}{2x^2}\right](dx-udt)-x^\nu pdx\right\}\equiv\\&\equiv\oint\left\{\left[\frac{u^2}{2}+h+\nu(2-\nu)\frac{\Gamma^2}{2x^2}\right]d\psi-Kx^\nu pdx\right\}=0\\&\oint d\psi\equiv K\oint x^\nu\rho(dx-udt)=0\end{aligned}\tag{1.9}$$

where the integration is carried out along an arbitrary closed contour in the xt, $x\psi$ or ψt plane. The terms with $\upsilon^2=\Gamma^2/x^2$ are important in (1.9) only when $\nu=1$, which, as in (1.3), reflects the factor $\nu(2-\nu)$. The laws of conservation in integral form (1.9) together with the integral laws of conservation of momentum (or the equivalent laws of conservation of angular momentum) are primary, while the differential equations (1.3) and (1.4) follow from them in subregions of continuity of the parameters.

For the subsequent analysis it is convenient to use the functions $R(u,p,s)$ and $L(u,p,s)$, which for $\nu=0$ and $s_0\equiv\text{const}$ are conserved along the c^+- and c^--characteristics, respectively, i.e. in this case they are invariants ("Riemann invariants" [5, 6]) of the characteristic system (1.7). In the general case we introduce R and L by means of the equations

$$2R=u+\Phi(p,s),\ 2L=u-\Phi(p,s),\ \Phi(p,s)=\int_{p_0(\psi)}^{p}\frac{dp}{\rho a}$$

where the integral defining $\Phi(p,s)$ is taken for constant $s=s_0(\psi)$, and consequently, ψ also. For $s_0\not\equiv\text{const}$ and $\nu\neq 0$ the functions R and L are not invariants. Despite this, by their definition

$$u=R+L,\ \Phi(p,s)=R-L\tag{1.10}$$

For fixed ψ, or what is the same thing for fixed s, by virtue of the definition of R, L and Φ and Eqs (1.2), the following expressions for the partial derivatives hold

$$\begin{aligned}&u_R=u_L=1,\ \Phi_R=-\Phi_L=1,\ p_\Phi=\rho a,\ p_{\Phi\Phi}=\frac{1}{2}\rho^4a^4\omega_{pp}\\&e_p=\frac{p}{\rho^2a^2},\ e_{pp}=\frac{1}{\rho^2a^2}-p\omega_{pp},\ h_{pp}=-\frac{1}{\rho^2a^2}\\&\rho_{pp}=\frac{2}{\rho a^4}-\rho^2\omega_{pp},\ a_p=-\frac{1}{\rho a}+\frac{1}{2}\rho^2a^3\omega_{pp}\end{aligned}\tag{1.11}$$

2. In general, the minimized functional (1.8) can be expressed in terms of the difference in the energies of the gas when $t = t_f$ and $t = 0$, using (1.9). As a result, apart from a constant term that is unimportant when solving the variational problem

$$A = \int_{f^\circ}^{f} \left[e + \frac{u^2}{2} + \nu(2-\nu)\frac{\Gamma^2}{2\xi} \right] d\psi \quad (\xi = x^{1+\nu}) \tag{2.1}$$

By virtue of equality (1.5), which introduces ψ, when $t = \text{const}$ we have $(\xi' = d\xi / d\psi)$

$$L \equiv \xi' - (1+\nu)/(K\rho) = 0 \tag{2.2}$$

In order to take this relationship into account, we set up the auxiliary functional

$$I = A + \int_{f^\circ}^{f} \mu(\psi) L d\psi$$

where the variable Lagrange multiplier $\mu(\psi)$ is to be determined. Any permissible variations δR, δL and $\delta\xi$, i.e. the difference between the initial and the varied values of R, L and ξ for fixed ψ, and, consequently, by (1.1) and (1.6), for fixed $p_0(\psi)$, $s(\psi)$ and $\Gamma(\psi)$, must satisfy the differential relationship (2.2). Hence, the variations of I and A are the same. Carrying out the necessary calculations and taking (1.2) and (1.11) into account we obtain

$$\delta A = (F_- - F_+)_d \Delta\psi_d + (\mu_- - \mu_+)_d \delta\xi_{d-} + \int_{f^\circ}^{f} \{ (u+\chi)\delta R + (u-\chi)\delta L -$$

$$- \left[\mu' + \nu(2-\nu)\frac{\Gamma^2}{2x^4} \right] \delta\xi + (\delta R)^2 + (\delta L)^2 - \frac{1}{4}\chi\rho^3 a^3 \omega_{pp} (\delta R - \delta L)^2 + \Omega(\delta\xi)^2 \Big\} d\psi \tag{2.3}$$

$$F = e + \frac{u^2}{2} - \frac{1+\nu}{K\rho}\mu_+, \quad \chi = \frac{p}{\rho a} + \frac{1+\nu}{K\rho a}\mu, \quad \Omega = \nu(2-\nu)\frac{\Gamma^2}{2x^6}$$

In deriving (2.3) we assumed that a discontinuity is possible at point d in the distributions of the parameters on $f^\circ f$. The parameters to the left (to the right) of d are given a minus (plus) sign, and we denote by $\Delta\psi_d$ the possible increment in ψ of the point of discontinuity. Since shocks when $t < t_f$ are forbidden by the formulation of the variational problem, discontinuities of the parameters on $f^\circ f$ can only be caused by focusing of similar characteristics at d. If there are several points of discontinuity, summation over all of them is assumed in (2.3).

For any (not necessarily optimal) distributions of the parameters on $f^\circ f$ the multiplier $\mu(\psi)$ using the arbitrariness of its choice, can be determined from the condition for the coefficient of $\delta\xi$ to vanish on $f^\circ f$, i.e. the requirement that in each section of continuity of the parameters

$$\mu' + \nu(2-\nu)\Gamma^2 / (2x^4) = 0 \tag{2.4}$$

Since (2.4) is a first-order differential equation, in addition to it we can put at the point (or points) of discontinuity of the parameters

$$\mu_{d+} = \mu_{d-} = \mu_d \tag{2.5}$$

i.e. make the multiplier μ continuous. Then expression (2.3) takes the form

$$\delta A = (F_- - F_+)_d \Delta\psi_d + \int_{f^\circ}^{f} \Big[(u+\chi)\delta R + (u-\chi)\delta L + (\delta R)^2 +$$

$$+ (\delta L)^2 - \frac{1}{4}\chi\rho^3 a^3 \omega_{pp} (\delta R - \delta L)^2 + \Omega(\delta\xi)^2 \Big] d\psi \tag{2.6}$$

In all cases when $\nu(2-\nu)\Gamma \equiv 0$, the multiplier μ, chosen from (2.4) and (2.5), is constant. The latter is natural because in these situations ξ does not occur explicitly in (2.1) for A. Hence, here, instead of the differential equation (2.2), we can take into account its corollary—the isoperimetric condition

$$X \equiv \xi_{f^\circ} - \xi_f = -\frac{1+\nu}{K}\int_{f^\circ}^{f}\frac{d\psi}{\rho} \tag{2.7}$$

This condition is included in the auxiliary functional $I = A + \lambda X$ by the constant Lagrange multiplier λ, which is identical with $\mu = \text{const}$.

Suppose now that $t_f = \tau_0$. Then the points of discontinuity may be the result of focusing of only the c^--characteristics, as shown in Fig. 1(a). In the plane isentropic case ($\nu = 0$, $s_0 = \text{const}$), for which an exact solution of the variational problem was obtained in [1, 2], in the triangle $af^\circ f$ one obtains a simple wave with $R \equiv 0$ and rectilinear c^--characteristics. Since here $\delta R \equiv 0$, $\Omega \equiv 0$ and $\xi = x$, we have from (2.3) and (2.4)

$$\delta A = (F_- - F_+)_d \Delta\psi_d + (\mu_- - \mu_+)_d \delta x_{d-} + \int_{f^\circ}^{f}\Big[(u-\chi)\delta L +$$

$$+(1 - \frac{1}{4}\chi\rho^3 a^3 \omega_{pp})(\delta L)^2\Big]d\psi \quad \left(F = e + \frac{u^2}{2} - \frac{\mu_+}{K\rho},\ \chi = \frac{p}{\rho a} + \frac{\mu}{K\rho a}\right) \tag{2.8}$$

with $\mu \equiv \text{const}$ at least on each section of continuity of the parameters. By specifying x_f and taking this fact into account using Eq. (2.2), integrated from fixed x_{f°, we obtain a method of determining these constants. To do this a "compensating" point k is introduced on each such section, at which, due to the arbitrariness of the choice of μ, the coefficient of δL in (2.8) vanishes. This gives

$$(\rho a)_k (u-\chi)_k \equiv (\rho a u - p - \mu / K)_k = 0$$

Now, by varying L in the neighbourhood of any point different from k of the corresponding section, we will simultaneously so vary L in the neighbourhood of k so as not to change the x coordinate of the right boundary of this section. Then, for fixed L on the other sections we ensure that all ψ_d and x_d and specified x_f are unchanged. Because of this the variations δL can be regarded as independent and, consequently, on each section of continuity of the parameters

$$\rho a(u-\chi) \equiv \rho a u - p - \mu / K = 0 \tag{2.9}$$

In the case considered the flow in $af^\circ f$ is a simple wave. Hence, the parameters of the gas in (2.9) are functions of one of them, for example, $u \equiv L$. Consequently, on each section of continuity of the parameters they are constant, and the optimum trajectory of the piston may consist only of sections of constant velocity and sections of acceleration, which ensure that c^--characteristics issuing from them on $f^\circ f$ are focused. We will show that the focal point of the characteristics is unique and that it coincides with f°, as shown in Fig. 1(b) (the x and t axes are not shown on it or henceforth). By taking the point that is "furthest to the right" we can calculate the coefficient of $\Delta\psi_d$ at this point from (2.8). Substituting μ_+, found from (2.9), into it, and temporarily dropping the subscript d, we obtain

$$\varphi \equiv F_- - F_+ = e_- + u_-^2/2 + (p - \rho a u)_+ / \rho_- - h_+ - u_+^2/2 + a_+ u_+ \tag{2.10}$$

This difference is a function of L_- and L_+, and is equal to zero when $L_+ = L_-$. By (1.2) and (1.11) we have

$$\frac{\partial\varphi}{\partial L_+} = 2a_+\left(1-\frac{\rho_+}{\rho_-}\right)\left(1-\frac{1}{4}\rho^3a^3\omega_{pp}u\right)_+$$

When the compression waves from the accelerating piston moving to the left are focused, $u_+<0$, $\rho_+/\rho_->1$, and $L_+\equiv u_+<L_-\equiv u_-$. Since $\omega_{pp}>0$, we hence obtain that $\partial\varphi/\partial L_+<0$, $(F_- - F_+)_d\equiv\varphi_d>0$ and for $\Delta\psi_d<0$ the variation $\delta A<0$. Negative values of $\Delta\psi_d$ are obtained for $x_d>x_{f^\circ}$ by a shift of the corresponding acceleration section of the piston "downwards" (i.e. by reducing the instant when the acceleration begins) with a slowing down of the piston (with respect to the modulus of the velocity) such that the point of intersection of the c^--characteristics issuing from the piston are shifted along $f^\circ f$ to the left. The preceding parts of the trajectory are then not varied. Hence, to the left of the discontinuity considered $\delta L\equiv 0$, $\delta x\equiv 0$, and consequently, at the point d investigated: $\delta x_{d^-}=0$. All the increments $\Delta\psi_d$ of the other points of discontinuity (if such points exist) are also equal to zero. The part of the trajectory close to f, although it is also corrected to ensure that the piston arrives at the specified final point, nevertheless, by virtue of (2.9), it does not make a contribution to δA that is linear with respect to δL. It can be shown that the contribution $(\delta L)^2$ in this way of variation has a higher order of smallness than $\Delta\psi_d$. It follows from this that our assertion is correct. In the case of Fig. 1(b) the piston, after acceleration, moves with a constant velocity. For such a trajectory, taking (2.9) into account we will have instead of (2.8)

$$\delta A = \varphi_d\Delta\psi_d + \int\limits_{d\equiv f^\circ}^{f}\left(1-\frac{1}{4}\rho^3a^3\omega_{pp}u\right)(\delta L)^2\,d\psi \tag{2.11}$$

with $\varphi_d>0$, where φ_d is calculated from (2.10), and $e_-=e_0$ while $u_-=0$. In (2.11) the point d coincides with f° and now the permissible $\Delta\psi_d$, unlike the internal focusing points, are non-negative. This is ensured by "raising" the acceleration section ab while simultaneously increasing the velocity of the piston (in modulus) along it. We will denote by $\delta^\circ L\equiv\delta^\circ u$ the variations of L and u along the trajectory when $t=\text{const}$. Then, using this method of varying the trajectory $\delta^\circ u>0$ in the neighbourhood of point a and, conversely, $\delta^\circ u<0$ when $t<t_b$ in the neighbourhood of point b, i.e. along ab the permissible $\delta^\circ u$ are not at all sign-constant, as was assumed in [1, 2]. When using (2.11) it is not the sign-constancy of $\delta^\circ u$ along ab that is important, but the non-negativity of $\Delta\psi_d$. Since, together with $\varphi_d>0$ in the problem considered $\omega_{pp}u<0$, then, by (2.11), any permissible variation of the constructed trajectory ($\Delta\psi_d\geqslant 0$, δL on $df\equiv f^\circ f$ are arbitrary) increases the work. Consequently, it is optimum.

The non-equivalence of the approaches based on the expressions for A in the form (1.8) and (2.1) is due to the fact that the original problem, "formulated on a trajectory", is, at first glance, only a problem of optimal control with ordinary differential equations. The point is that the condition for there to be no shock waves on $af^\circ f$ for a flow described by partial differential equations cannot be replaced here by a simple limitation on the velocity of the piston (or on the sign of $\delta^\circ u$) on the acceleration part of the trajectory. Such a situation is typical for gas dynamics where different forms of writing the optimized functionals usually supplement one another.

3. In the light of the last statement, we will begin our consideration of the case $t_f>\tau_0$ by formulating the problem of the required trajectory for a plane piston and $s_0\equiv\text{const}$. We will take the work A in the form (1.8), and the isoperimetric condition is the specification of the difference in the coordinates of the piston

$$X\equiv x_a-x_f=-\int\limits_a^f u\,dt$$

and we will introduce into the auxiliary functional $I=A+\lambda X$ a constant Lagrange multiplier

λ. Suppose (Fig. 1c, cd is the c^+-characteristic) that the specified t_f is such that $x_d \leqslant x_f$. Then, in $acdf$ we will have a simple wave with $R \equiv 0$ and $L = u$, and taking this into account we obtain

$$\delta A = K\int\limits_a^f\left[\left(\rho au - p - \frac{\lambda}{K}\right)\delta^{\circ}u + \rho a\left(1 - \frac{1}{4}\rho^3 a^3 \omega_{pp} u\right)(\delta^{\circ}u)^2\right]dt \tag{3.1}$$

Omitting for the moment the considerations in Sec. 2 on the formulation of the conditions for no shock waves when $t < t_f$, by analysing (3.1) we can obtain the optimum trajectory which the flow achieves, shown in Fig. 1(d). The part on which the piston moves with constant velocity adjoins the acceleration part, as in Fig. 1(b). Now, however, the c^+-characteristics are focused at point d, and depart from the walls as a reflection of the c^--characteristics, arriving from ab. In $bgdf$ the gas parameters are constant, and in particular $u \equiv \text{const} < 0$, while in the triangle $ef^{\circ}d$ the gas is at rest, being slowed down to a simple compression wave edg. On bf

$$\varphi(u) \equiv \rho au - p - \lambda / K = 0 \tag{3.2}$$

and on ab: $\varphi > 0$. The latter is a consequence of the fact that, by (1.2) and (1.11)

$$d\varphi / du = 2\rho a\left(1 - \frac{1}{4}\rho^3 a^3 \omega_{pp} u\right) > 0$$

$\varphi_b = 0$ and $u > u_b$ when $t < t_b$. By virtue of this inequality on bf there cannot be "internal acceleration sections" such that the c^--characteristics running from them are focused on $f^{\circ}f$ without reflection or with reflection from the walls.

Condition (3.2) can be obtained in the same way as the similar equation (2.9), apart from the replacement of $f^{\circ}f$ on the trajectory af. Taking the compensating point on qf, one must vary the trajectory in all allowable ways. One can, for example, in addition to the neighbourhood k change u in the neighbourhood of only one point of the acceleration section. In this variation $\delta^{\circ}u$ are positive on ab, because for $\delta^{\circ}u < 0$ part of the c^+-characteristics of the pencil would intersect close to d when $t < t_f$. For $\delta^{\circ}u > 0$ on ab the positiveness of φ on the acceleration part gives $\delta A > 0$, which is not unlike the way in which the optimality of the trajectory constructed is proved. Unfortunately, however, here, as in the analysis in [1, 2], for $t_f = \tau_0$, the possibility that $\delta^{\circ}u < 0$ on ab in the variation with a "rise" of ab, described in Sec. 2, is not taken into account. On the other hand, for an arbitrary variation of $u = L$ on bq, perturbations proceeding along the c^--characteristics deform the pencil of c^+-characteristics, which may cause them to intersect when $t < t_f$. Consequently, variations of $\delta^{\circ}u$ on this section are not arbitrary and in this sense condition (3.2), which makes the linear term in (3.1) vanish, is excessively severe. When $\nu = 0$ and $s_0 \equiv \text{const}$ the latter, it is true, unlike the general case (Sec. 6) does not impair the solution constructed. These considerations, and also the desire to carry over the scheme shown in Fig. 1(d) to the general case, justify investigating it by transferring to the section $f^{\circ}f$.

After transferring to $f^{\circ}f$, i.e. to expression (2.6) for the flow with $u \equiv 0$ on $f^{\circ}d$ for any ν, s_0 and Γ we obtain

$$\begin{aligned}\delta A = (F_- - F_+)_d \Delta\psi_d + \int\limits_{f^{\circ}}^{d}\Big\{\chi\Big[1 - \frac{1}{4}\rho^3 a^3 \omega_{pp}(\delta R - \delta L)\Big](\delta R - \delta L) + \\ +(\delta R)^2 + (\delta L)^2 + \Omega(\delta\xi)^2\Big\}d\psi + \int\limits_d^f\Big[(u+\chi)\delta R + (u-\chi)\delta L + (\delta R)^2 + \\ +(\delta L)^2 - \frac{1}{4}\chi\rho^3 a^3 \omega_{pp}(\delta R - \delta L)^2 + \Omega(\delta\xi)^2\Big]d\psi\end{aligned} \tag{3.3}$$

In the plane isentropic case, when $\Omega \equiv 0$, while $\delta R \equiv R \equiv 0$ on df, as in Sec. 2, $L \equiv u \equiv \text{const}$ on

df, and is defined in the triangle dfq by condition (2.9) with $\mu \equiv \text{const}$. As a result, expression (3.3) is simplified somewhat

$$\delta A = (F_- - F_+)_d \Delta\psi_d + \int_{f^\circ}^{d} \left\{ \chi\left[1 - \frac{1}{4}\rho^3 a^3 \omega_{pp}(\delta R - \delta L)\right](\delta R - \delta L) + \right.$$

$$\left. +(\delta R)^2 + (\delta L)^2 \right\} d\psi + \int_{d}^{f} \left(1 - \frac{1}{4}\rho^3 a^3 \omega_{pp} u\right)(\delta L)^2 d\psi \tag{3.4}$$

In the flow investigated $u \equiv 0$, $a \equiv a_f$, in $ef^\circ d$, while in the pencil edg all the c^+-characteristics are rectilinear. On reflection from the wall, where $u = R + L = 0$, the perturbations R are reflected by the perturbations L of opposite sign. Taking these facts into account it can be shown that the same variations of L or $-R$ in the ε-neighbourhood of an arbitrary point $f^\circ d$, which we will represent by the quantity

$$\sigma = \int_{\psi-\varepsilon}^{\psi+\varepsilon} \delta L d\psi \vee - \int_{\psi-\varepsilon}^{\psi+\varepsilon} \delta R d\psi$$

cause the same deformation of the pencil. When making an analysis with a transfer to $f^\circ f$, the discontinuity at point d is shifted, without being spread out (if the optimum distribution of the parameters on $f^\circ f$ is continuous, the discontinuity is not introduced from the beginning!). Hence, the spreading out of the pencil connected with $\sigma \neq 0$ should be compensated by a correction of the initial part of the trajectory such that the intersection of all the characteristics of the pencil occurs for a specified $t = t_f$. It can be shown that this causes a shift of the beam proportional to σ. In addition to this, independent variation of the acceleration part with a "rise" and a shift of d to the left is permissible. Consequently, $\Delta\psi_d = \Delta\psi_{d0} + N\sigma$, where N is a certain coefficient which depends on the flow in the pencil edc, while $\Delta\psi_{d0} \leq 0$.

We recall that "consideration on the trajectory" only left open the question of the variation with a "rise" of its initial part, which, in the expression derived above for $\Delta\psi_d$, corresponds to $\Delta\psi_{d0}$. Bearing only this fact in mind, instead of (3.4) we obtain

$$\delta A = (F_- - F_+)_d \Delta\psi_{d0} + \dots \tag{3.5}$$

Here the dots denote terms due to $\sigma \neq 0$ on $f^\circ d$ and $(\delta L)^2$ on df. When solving the problem of interest to us it is sufficient to determine the sign of $\varphi = (F_- - F_+)_d$, where now, unlike (2.10) $\varphi = \varphi(R_-, R_+)$, but like (2.10) $\varphi(R_-, R_-) = 0$. Taking (2.9) into account and the fact that in the case considered $u_- = 0$, and temporarily omitting the subscript d, we obtain

$$\varphi = h_- - h_+ - \frac{u_+^2}{2} + \left(1 - \frac{\rho_+}{\rho_-}\right) a_+ u_+ + \frac{p_+ - p_-}{\rho_-} \tag{3.6}$$

In the case investigated $p_+ < p_-$, $\rho_+ < \rho_-$ and $u_+ < 0$, by virtue of which all the terms on the right-hand side of (3.6), apart from the first, are negative. This, of course, is insufficient for the negativeness of φ, in particular $h_- - h_+ > 0$. Recalling that $\varphi(R_-, R_-)$ and $R_+ < R_-$, we obtain

$$\frac{\partial\varphi}{\partial R_+} = -\left[2 + \frac{1}{2}\left(\frac{\rho}{\rho_-} - 1\right)\rho^3 a^4 \omega_{pp}\right]_+ u_+ = -\left[2 + \frac{\kappa+1}{2}\left(\frac{\rho_+}{\rho_-} - 1\right)\right] u_+ > \frac{\kappa - 3}{2} u_+$$

where the second equality and inequality hold for an ideal gas with adiabatic index κ. Hence, at least for an ideal gas with $\kappa \leq 3$, it necessarily follows that $\varphi < 0$ and, by (3.5) $\delta A \geq 0$ when $\Delta\psi_{d0} \leq 0$. Here, as in Sec. 2, it can be shown that when there is a "rise" of the acceleration section of the trajectory, terms denoted by dots in (3.5) are of a higher order of smallness than $\varphi\Delta\psi_{d0}$.

4. For $\nu=0$ and $s_0\equiv\text{const}$ the set of formulations of the variational problem on the trajectory and in the section $t=t_f$ enables us to prove that the trajectories which give the scheme of the flow in Fig. 1(b) and (d), in fact ensure a minimum of A. For arbitrary ν, s_0 and Γ one cannot carry out such a complete analysis as is typical for variational problems in gas dynamics. Moreover, the formulation on the trajectory in general does not occur, since the solution with $R\equiv 0$ (a "simple wave") is not justified. The possibilities of transferring to the section $t=t_f$ are also reduced. On the other hand, the requirement that there should be no shock waves when $t<t_f$ indicates that the schemes shown in Fig. 1(b) and (d) (naturally with $u\not\equiv\text{const}$ on bf and with curvilinear characteristics of the pencils) where "checked for optimality" for any ν, s_0 and Γ. This "check" is particularly simple in the special case of the scheme shown in Fig. 1(d), when points d and f coincide (Fig. 1e). A similar situation may occur when $t_f\geqslant\tau_m$, where the previously mentioned time τ_m is determined during the solution.

Basing on (3.3), i.e. on the transition to $f^\circ f$, it can be shown that when $t_f\geqslant\tau_m$ in general the solution with $u\equiv 0$ is optimal on $f^\circ f$, and as a consequence in $ef^\circ f$. When $u\equiv 0$ on $f^\circ f$ (3.3) reduces to

$$\delta A=\varphi\Delta\psi_d+\int_{f^\circ}^{f}\Big\{\chi\Big[1-\frac{1}{4}\rho^3a^3\omega_{pp}(\delta R-\delta L)\Big](\delta R-\delta L)+$$

$$+(\delta R)^2+(\delta L)^2+\Omega(\delta\xi)^2\Big\}d\psi \tag{4.1}$$

where permissible values of $\Delta\psi_d\leqslant 0$, when calculating $\varphi=(F_--F_+)_d$ from (2.5) $\mu_+=\mu_-$, while the parameters with the plus and minus subscripts are related by the conditions: $L_+=L_-$ and $R_+<R_-$. These conditions correspond to a variation of the trajectory with a "rise" of its acceleration section, when the focal point of the c^+-characteristics is shifted to the left along $f^\circ f$. If $t_f>\tau_m$, such a way of variation is possible for which the pencil of c^--characteristics reflected from the varied trajectory arrives at point d. In this case at d: $R_+=R_-$, while $L_+<L_-$. For any permissible way of variation $\Delta\psi_d\leqslant 0$ and $\chi(\delta R-\delta L)$ is a unique linear term which is not prevented from being sign-variable. Hence, by investigating the flow scheme shown in Fig. 1(e) with $u\equiv 0$ on $f^\circ f$ for a minimum of A, it is natural to equate the factor χ to zero and to consider that this is given. Thus, let us suppose that on $f^\circ f$

$$\rho a\chi\equiv p+(1+\nu)\mu/K=0 \tag{4.2}$$

We recall that the multiplier μ that occurs here, apart from the specification, for example, of μ_{f°, is defined by differential equation (2.4). If, according to (4.2), we put $\mu_{f^\circ}=-Kp_{f^\circ}/(1+\nu)$, then (2.4) and (4.2) define μ and p over the whole section $f^\circ f$. Differentiating (4.2) with respect to x and eliminating $\mu_x=\mu'\psi_x=Kx^\nu\rho\mu'$ using (4.2) we obtain that on $f^\circ f$

$$\frac{\partial p}{\partial x}=\nu(2-\nu)\frac{\rho\upsilon^2}{x} \tag{4.3}$$

i.e. either $p\equiv\text{const}$ or (for $\nu=1$ and $\Gamma\not\equiv 0$) the flow is radially in equilibrium. By (1.3) both this and the other ensures that $u\equiv 0$ and $p\equiv\text{const}$, and for $\nu=1$ and $\Gamma\not\equiv 0$ it ensures radial equilibrium of the flow over the whole triangle $ef^\circ f$.

By virtue of (4.2), instead of (4.1) we obtain

$$\delta A=\varphi\Delta\psi_d+\int_{f^\circ}^{f}[(\delta R)^2+(\delta L)^2+\Omega(\delta\xi)^2]d\psi \tag{4.4}$$

Since $\Omega\geqslant 0$ and $\Delta\psi_d\leqslant 0$, it remains to determine the sign of φ for the ways described above for varying the trajectory. Expressing μ in terms of p_- from (4.2) and, as previously, omitting the subscript d, we obtain that now

$$\varphi = h_- - e_+ - \frac{p_-}{\rho_+} - \frac{u_+^2}{2}, \ \frac{\partial\varphi}{\partial R_+} = -u_+ + \frac{p_- - p_+}{\rho_+ a_+} > 0$$

$$\frac{\partial\varphi}{\partial L_+} = -u_+ + \frac{p_+ - p_-}{\rho_+ a_+} > 0$$

Here $\partial\varphi/\partial R_+$ is determined for the situation in which the pencil of c^+-characteristics is shifted to the left, while $\partial\varphi/\partial L_+$ is determined for the pencil of the c^--characteristics emerging from the trajectories. In both cases, $u_+ < 0$, but in the first of these $p_- > p_+$, while in the second, on the contrary, $p_- < p_+$. Moreover, as already pointed out, $\varphi(R_-, R_+)$ and $\varphi(L_-, L_+)$ vanish for $R_+ = R_-$ and $L_+ = L_-$, respectively, while $R_+ < R_-$ and $L_+ < L_-$. Hence, in both situations $\varphi < 0$ and for any permissible variation of the trajectory of the piston ($\Delta\psi_d \leqslant 0$, the signs of δR, δL and $\delta\xi$ on $f^\circ f$ are arbitrary) δA is non-negative by virtue of (4.4). Consequently, the solution obtained gives a minimum of A.

5. The practical construction of the optimum trajectory corresponding to the scheme shown in Fig. 1(e) and the determination of the interval τ_m reduces to the numerical solution of the following problems. Initially, using the specified distributions of the parameters at $t = 0$ the overall mass of gas

$$M = \int_{a^\circ}^{a} x^\nu \rho_0(x)dx$$

is calculated. The normalizing factor K is equal to $1/M$. With this choice of K and $\psi_{a^\circ} = 0$ on the piston: $\psi_a = 1$. Then for the same initial distributions from (1.5) with $dt = 0$ one calculates $\psi = \psi_0(x)$, and as a result of this, the right-hand side of (1.6) and $p_0(\psi)$ satisfying the condition of radial equilibrium (1.1) are determined. Then the parameters of the gas on $f^\circ f$, differing from $u \equiv 0$, are found using (1.6) by numerical integration of Eq. (2.2), in which $\rho = \rho[p, s_0(\psi)]$, and of radial equilibrium equation (4.6). It is convenient to use the latter in the following form

$$p' \equiv \frac{\partial p}{\partial\psi} = \nu(2-\nu)\frac{\Gamma^2}{K\xi^2}$$

The arbitrariness in choosing the pressure $p_{f^\circ} > p_0(0)$ is used here to satisfy the condition: $\xi(1) = \xi_f = x_f^{1+\nu}$ for specified $x_f < x_a$. As already noted, the radial equilibrium flow calculated in this way (or for $\nu(2-\nu)\Gamma \equiv 0$ — a stationary gas, in which $p \equiv p_{f^\circ}$) is preserved over the whole triangle $ef^\circ f$, bounded below by the c^+-characteristic ef. Further calculation is carried out from the section $f^\circ f$ in the direction of decreasing time, the origin of which at this stage is conveniently taken to coincide with this section by putting $t_f = 0$. The characteristic ef is constructed by integrating the second equation of system (1.7) with a plus sign and ρa is a known function of ψ. The integration is carried out from the point f at which $\psi = 1$ and $t = 0$. The dependence of x on ψ along ef is the same as along $f^\circ f$. The "condition of compatibility" (the third equation of system (1.7)), which for $u \equiv 0$ in $ef^\circ f$ reduces to the condition of radial equilibrium, is satisfied automatically.

From the data on ef and the tangency condition $u = 0$ on the fixed wall ($\psi = 0$, $x = x_{f^\circ}$, $t < t_e$) the pencil of c^+-characteristics is calculated (when the calculations are carried out in the direction in which t decreases—the pencil of rarefaction waves) to the point c where the pressure, falling continuously as time decreases, becomes equal to the previously known value of $p_0(0) < p_e = p_{f^\circ}$. The construction of the "initial" c^--characteristic ca, which bounds the stationary or radially equilibrium gas in the triangle $a^\circ ca$, leads similarly to the construction of the c^+-characteristic ef with the already known factor K and the dependences $\psi = \psi_0(x)$ or $x = x_0(\psi)$ which hold on ca. As a result of this calculation the time $t_a < 0$ is found, and from it also $\tau_m = -t_a > \tau_0$ (in typical situations $\tau_m > 2\tau_0$). After determining t_a by changing the origin of coordinates t, solving the Goursat problem with data on the characteristics fc and ca and determining from it the line af, on which $\psi = 1$, we obtain the trajectory of the piston. The value of τ_m

obtained, is the minimum time for which the shock-free compression of the stationary or radially equilibrium gas to the same state with high average density is possible. Guaranteeing this compression of the piston trajectory is unique. For $t_f > \tau_m$ this problem has an infinite set of solutions.

Using the same method, taking $\psi_d < 1$ and $x_d < x_0(\psi_d)$, one can calculate the flow, shown in Fig. 1(d,) up to the c^--characteristic qd and obtain the part of the trajectory aq. However, in general $\nu \neq 0$ or $s_0 \neq \text{const}$, when $u \neq \text{const}$ on bf, the data on the characteristic qd obtained in this way is insufficient to construct the final part of the trajectory qf. The position for the scheme shown in Fig. 1(b) is similar with the sole difference that here the calculation by the method of characteristics with $\nu \neq 0$ and $x_{a^\circ} = 0$ must be supplemented by the self-similar solution describing the focusing of the characteristics at f°. In this scheme, after determining the flow on the c^--characteristic af° and calculating the pencil of compression waves from the point f° and the trajectories beginning at the point a $(\psi = 1)$ one need only construct its acceleration part of previously unknown extension. To construct the end section of the optimum trajectory in the case shown in Fig. 1(b) and (d) with $\nu \neq 0$ or $s_0 \neq \text{const}$ additional information is necessary

6. The information necessary to construct $\nu \neq 0$ or $s_0 \neq \text{const}$ of the final parts of the optimum trajectory in the schemes shown in Fig. 1(b) and (d), is obtained, as mentioned, by the method of undetermined control contour (MUCC). To do this, by (1.9), we express A in terms of the integral over the as yet undetermined but fixed control contour alf. The integral by parts of the contour lying below the c^--characteristics af° and ac does not change when the trajectory varies. Hence, when solving the variational problem only the section lf is important. If lf in the ψx plane is specified by the equation $x = x(\psi)$ and its corollary $x' = x'(\psi)$, we obtain the following equation for A taking this into account

$$A = \int_l^f \left[h + \frac{u^2}{2} + \nu(2-\nu)\frac{\Gamma^2}{2x^2} - Kx^\nu x' p \right] d\psi$$

Similarly, the specification of t_f by virtue of (1.5) is equivalent to constancy of the integral

$$\tau \equiv t_l - t_f = \int_l^f \left(\frac{1}{Kx^\nu \rho u} - \frac{x}{u} \right) d\psi$$

For $\psi_d < \psi_f$ and alf, which with $f^\circ f$ has a unique common point f, the parameters of the gas on lf are continuous both for the trajectory investigated and for the varied trajectory. Hence, by setting up an auxiliary functional $I = A + \lambda\tau$ with constant Lagrange multiplier λ, we obtain, after necessary calculations, that

$$\delta A = \delta I = \int_l^f [r(1-\Lambda)\delta R + l(1+\Lambda)\delta L + (g+f)(\delta R)^2 + (g-k)(\delta L)^2 + $$
$$+ 2(1 - \Lambda - g + f)\delta R \delta L] d\psi \tag{6.1}$$

$$r = u + a - x' K x^\nu \rho a, \quad l = u - a + x' K x^\nu \rho a$$

$$g = (r - u + \Lambda u)\rho^3 a^3 \omega_{pp} / 4, \quad f = r\Lambda / u$$

$$k = l\Lambda / u, \quad \Lambda = \lambda / (K x^\nu \rho a u^2)$$

Using the arbitrariness in the choice of the undetermined control contour or, which is the same thing, the function $x'(\psi)$, the factor r in front of δr vanishes. As a result we obtain

$$x' = (u + a) / (K x^\nu \rho a) \tag{6.2}$$

i.e. by (1.7) lf is the section of the c^+-characteristic. The vanishing of the coefficient in front of the linear term remaining after this in (6.1) gives the necessary condition for an extremum: $\Lambda = -1$, or, taking into account the expression for Λ

$$x^{\nu}\rho a u^2 = \text{const} = -\lambda / K \tag{6.3}$$

This condition cannot be satisfied over the whole characteristic lf, in particular at the point l where $u = 0$, because in this case, from (6.3) $u \equiv 0$ on lf. The latter is possible in the special case of a fixed piston, which is of no particular interest. In the case of the scheme shown in Fig. 1(b) and (d), only the end sections of the trajectory of the piston, which corresponds to the section hf of the characteristic lf, remains to be constructed. Hence it is natural to use (6.3) only on hf, thereby closing the problem of constructing the whole required trajectory. At the same time, for $\tau_0 < t_f < \tau_m$ (Fig. 1d) it remains tempting to extend (6.3) to the section nh, especially as, when $\nu = 0$ and $s_0 \equiv$ const, this, like the "formulation of the problem on a trajectory", gives $u \equiv$ const everywhere in $bgdf$ and $u \equiv 0$ in $ef°d$. An attempt to extend (6.3) to nh does not, however, pass the test of being the optimum solution for $t_f = \tau_m$ (Fig. 1e). In fact, for $t_f \to \tau_m - 0$ the section nh becomes the part af (Fig. 1e) of the "closing" characteristic of the pencil of c^+-characteristics, and (6.3) is certainly not justified on it. The deep reason for the non-validity of extending (6.3) to nh is the non-arbitrariness, pointed out in Sec. 3, of the variation of u along the part bq of the trajectory of the piston and as a consequence of this the variation of L along nh.

Taking (6.2) and (6.3) into account as well as the above discussion, expression (6.1) for δA takes the form

$$\delta A = 2\int_l^h u(1+\Lambda)\delta L d\psi + \int_h^f [-\frac{1}{2}\rho^3 a^3 \omega_{pp} u(\delta R)^2 + 2(1 - \frac{1}{4}\rho^3 a^3 \omega_{pp} u)(\delta L)^2 + \\ +(4 + \rho^3 a^3 \omega_{pp} u)\delta R \delta L] d\psi \tag{6.4}$$

In the integral over lh, where $1 + \Lambda \not\equiv 0$, the quadratic terms are unimportant.

When only the end section of the trajectory qf is varied, when $\delta L \equiv 0$ along lh, it can be shown that δR has a higher order of smallness along hf than δL. Hence it follows that the necessary condition for the section to be optimal (a minimum of A) is the inequality $1 - \rho^3 a^3 \omega_{pp} u/4 > 0$, which is necessarily satisfied in the problem considered in which $\omega_{pp} u < 0$.

When analysing the contribution of the integral over lh from (6.4) the sign of $(1+\Lambda)$ is important. Substituting the constant λ into the formula for Λ, expressed in terms of the parameters in h by (6.3), we obtain

$$1 + \Lambda = 1 - (x^{\nu}\rho a u^2)_h / (x^{\nu}\rho a u^2)$$

By virtue of (1.11) and (1.7) along any c^+-characteristic

$$\frac{d}{dt}(x^{\nu}\rho a u^2) = x^{\nu}\rho a\left(1 - \frac{1}{4}\rho^3 a^3 \omega_{pp} u\right)\frac{du^2}{dt} + \nu x^{\nu}\rho a u^2\left[a - \left(\frac{1}{2}\rho^3 a^4 \omega_{pp} - 1\right)u\right] + \\ + \frac{\nu(2-\nu)}{2} x^{\nu-1}\rho^4 a^4 \omega_{pp} u^2 \upsilon^2 \tag{6.5}$$

Since $u \leqslant 0$, for $\nu \neq 0$ the second and third terms on the right-hand side of (6.5) are positive (although for the second term this assertion is possibly categorically unnecessary, in the case of an ideal gas with $\kappa \geqslant 1$ this is necessarily so). Along hf the left-hand side of (6.5) is zero. Consequently, when $\nu \neq 0$ along hf the velocity of the gas is reduced in absolute value. Along the acceleration sections (along lh and along bn, respectively, for Fig. 1b and d) u^2 increases, and of course, $x^{\nu}\rho a u^2$ also increases. In the case of Fig. 1(b) it therefore follows immediately that along lh: $1 + \Lambda \leqslant 0$ and $u(1+\Lambda) \geqslant 0$. If the trajectory of the piston is varied without a "rise" of its acceleration part, it can be shown that for Fig. 1(b) in the most general case along lh one can have $\delta L \geqslant 0$ and consequently $\delta A > 0$. Variation with a "rise" of the acceleration section and the more complex case shown in Fig. 1(b) require additional investigations.

For $s_0 \equiv \text{const}$ and $\Gamma \equiv \text{const}$ we can apply the method of undetermined control contour in the xt plane also. Here, instead of (6.3), to determine the section hf of the extremal c^+-characteristic we obtain

$$2h + u^2 - 2au + \nu(2-\nu)\Gamma^2 x^{-2} = \text{const} \tag{6.6}$$

The condition (6.3), obtained for arbitrary s_0 and Γ also holds along hf. As in variational problems of steady supersonic flows [3], the compatibility condition for the c^+-characteristics follows from (6.3) and (6.6) with $s_0 \equiv \text{const}$ and $\Gamma \equiv \text{const}$, i.e. in this case on the extremal characteristic (6.6)—the integral (1.7). It has already been pointed out that (6.3) and (6.6) with $\Gamma \equiv 0$ are identical to the conditions defining the "extremal" characteristic of the problem of the optimum expansion of the piston, solved for $\Gamma \equiv 0$ in [4].

For $\nu = 0$ and $s_0 \equiv \text{const}$, when the schemes shown in Fig. 1(b) and (d) in addition to the compression waves, by the construction of the focusing for $t = t_f$, contain only regions of constant parameters, the absence of shocks for $t < t_f$ is guaranteed. The construction of the optimum trajectory in the case of Fig. 1(e) after inverting t is equivalent to the problem of profiling a supersonic nozzle which converts one parallel axis of the flow into the same flow with a higher velocity. Experience in profiling such nozzles does not reveal the occurrence when calculating them of a "gradient catastrophe". The position with the structure of the "optimum" flow from the left from the characteristics bf° and qd in Fig. 1(b) and (d) is similar. The remaining open question (for $\nu \neq 0$ and $s_0 \not\equiv \text{const}$) of whether a gradient catastrophe is possible from the right of these characteristics can only be solved by specific calculations.

Above, shock waves and the increase in s when $t < t_f$ prevented the problem being formulated. On the other hand, this obstacle may also result from physical considerations, according to which an increase in s represents additional losses, and is just like an increase in A. We will consider how one can verify these considerations in analysing the expression for δA, including an increase in s at the jumps that occur when $t < t_f$. For the scheme shown in Fig. 1(e) if we assume that $s = s_0(\psi) + \delta s$ since $\delta s \geqslant 0$, then the following additional term will occur in δA

$$\delta A = \ldots + \int_{f^\circ}^{f} T\delta s d\psi \tag{6.7}$$

where the dots denote the right-hand side of (4.4). When obtaining (6.7) we took into account the fact that $h_s \equiv (\partial h/\partial s)_p = T$, and by virtue of (4.2) $\chi = 0$ on $f^\circ f$. When $\delta s > 0$ the term δs in (6.7) in fact increases A. For Fig. 1(b) and (d), i.e. for $t_f < \tau_m$, the above-mentioned "physical considerations" are not in general true. For example, for $\nu = 0$ and $s_0 \equiv \text{const}$ one can so increase the velocity of the piston along ab that the jumps which occur in the compression wave when $t < t_f$ do not change $R \equiv 0$ along af. In this case A decreases.

All the above discussion can be transferred to the case $x_a < x_{a^\circ}$ when the piston and the wall change places (when $\nu \neq 0$ the wall becomes "external", and the piston expands "inwards"). The flow patterns, which will therefore change the schemes shown in Fig. 1(b) and (d), are shown in Fig. 1(f) and (g). The main difference in this case is the fact that the velocity u is positive and the roles of the c^+- and c^--characteristics are interchanged. In particular, (6.3) is now satisfied along hf—the section of the c^--characteristic, and instead of (6.6) with $s_0 \equiv \text{const}$ and $\Gamma \equiv \text{const}$ along hf in addition to (6.3) the following relation holds

$$2h + u^2 + 2au + \nu(2-\nu)\Gamma^2 x^{-2} = \text{const}$$

It is more fundamental to replace the walls by a second piston. This problem for $t_f > \tau_0$ and an arbitrary position of the points f and f° cannot be reduced to the superposition of the schemes shown in Fig. 1(d) and (g).

7. We will compare the work A_0 expended in optimum compression when $t_f = \tau_0$, $\nu = 0$ and $s_0 \equiv \text{const}$, with the work A_m which is required for the optimum compression of the same gas when

$t_f = \tau_m$ in the scheme shown in Fig. 1(e), for which the result is independent of ν (for $\nu = 1$ — when there is no twisting). If we take the initial density and velocity of sound as the scales of density and velocity, we have $\rho_0 = a_0 = 1$ and $e_0 = 1/[\kappa(\kappa-1)]$. In the cases compared the parameters of the gas for $t = t_f$ are independent of x. Hence, by finding A in terms of the difference in energies for $t = t_f$ and $t = 0$, we obtain, for an ideal gas

$$A_0 = \frac{(3\kappa-1)a_f^2 - 4\kappa a_f + \kappa + 1}{\kappa(\kappa-1)^2} M, \quad A_m = \frac{a_f^2 - 1}{\kappa(\kappa-1)} M$$

$$E_m = \frac{a_f^2 M}{\kappa(\kappa-1)}, \quad \alpha \equiv \frac{A_0}{A_m} = \frac{(3\kappa-1)a_f^2 - 4\kappa a_f + \kappa + 1}{(\kappa-1)(a_f^2-1)} \tag{7.1}$$

$$\eta \equiv \frac{A_m}{E_m} = \frac{a_f^2 - 1}{a_f^2}, \quad a_f = \rho_f^{(\kappa-1)/2}$$

Here M is the mass of compressed gas, and $E_m = Me$ is its total energy, and when obtaining A_0 we used the relationship between u_f and a_f for a simple wave with $R \equiv 0$. By virtue of (7.1) $\alpha \to 1$ when $\rho_f \to 1$ and $\alpha \to (3\kappa-1)/(\kappa-1)$ when $\rho_f \to \infty$, i.e. for infinite compression of the gas. In fact, however, despite the extremely simple relationship between α and a_f and between a_f and ρ_f, the approach of α to its limiting value for real $\kappa \leqslant 1.4$ as ρ_f increases occurs extremely slowly. This is shown in Fig. 2, in which for different values of κ the continuous curves give the values of α as a function of lg ρ_f. It can be seen that, for example, for $\kappa = 1.1$ even with $\rho_f = 10^{10}$ the ratio of the works is far from its limiting value. Nevertheless, a considerable gain occurs for compression after a time $t_f = \tau_m$ even for medium ρ_f. For $1 \leqslant \rho_f \leqslant 9$ this can be clearly seen from Fig. 3. The dashed curves and the scale on the right in Figs 2 and 3 represent η—the fraction of the work in the final energy of the gas.

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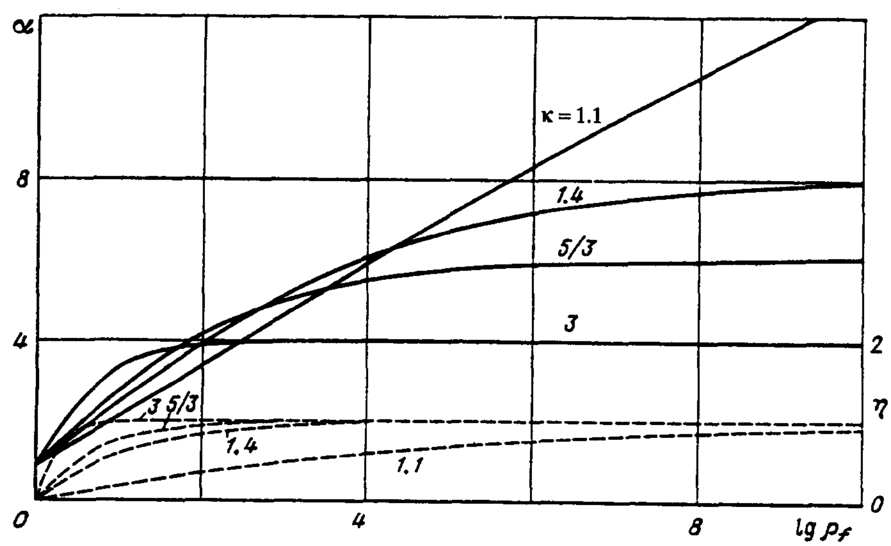

FIG. 2.

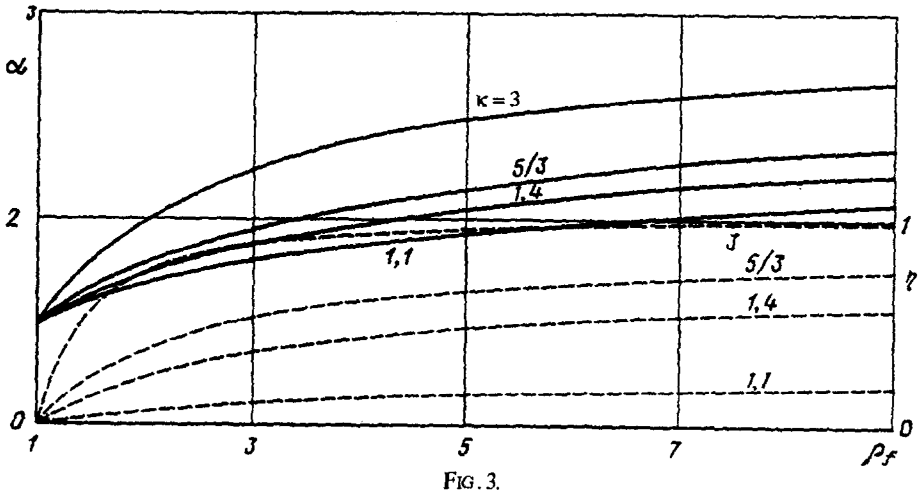

FIG. 3.

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